The Structure of Classical Value Theories

I. INTRODUCTION

The labour theory of value seems remote from contemporary economics. Yet it has more than an historical interest for economists. It is, in a modified form, the doctrine upon which economic decisions are based in important sectors of the world. Moreover, in recent years there has been an increased interest in certain aspects of Marxist political economy, an interest stimulated apparently by comparisons with Keynesian and other current theories.[1] Yet this interest has not extended to an analysis of the value theory which Marx considered essential to his economics, so that there exists no sound basis for interpretation of the latter. Such considerations impelled the writer to attempt a precise formal re-statement of the labour theory of value as found in the words of Adam Smith and Kark Marx.[2] This paper contains definitions of "value" equivalent to those found in *The Wealth of Nations* and in *Capital*, Vol. I, and of "price of production" equivalent to that given in *Capital*, Vol. III. The definitions are used to comment on various controversial issues, to indicate links between classical and contemporary theory, and to give a solution to the so-called "transformation problem."

II. A VALUE THEORY OF ADAM SMITH

In its crudest form the labour theory states that the value of a commodity is given by the amount of labour involved in its production. This is to be measured not alone by the amount of direct labour but also by the labour embodied in other factors of production. The latter is to be measured in the same way by reference to the production of the factors, and so on. The "and so on" suggests either an infinite regression or some type of circularity. In order to see what meaning, if any, can be assigned to such ideas, we will formulate mathematically a value concept which appears in Chapters V–VIII of *The Wealth of Nations*. We say "a value concept" because Smith actually mingles several partially inconsistent notions.[3]

Consider a closed economy which has been operating during a certain time interval. Let q_i represent the quantity of the ith commodity produced during the time interval, where goods are classified so that capital goods correspond to $i = 1, 2, \ldots n$ and consumption goods to $i = n + 1, \ldots n + m$. Goods which play a role both as capital and consumption goods will be divided for accounting purposes. Let q^i_j represent the amount of j used in producing q_i, where j runs only from 1 to n. These factor amounts represent goods used up, physical depreciation rather than stocks in the case of fixed capital goods. The sum $\sum_{i=1}^{n+m} q^i_j$ will equal q_j only under static assumptions. Let h_k be the amount of labour of type k, measured in hours of type k, worked during the interval, and let h^i_k be the amount of labour of this type used directly in producing q_i. Finally, let q^k_i, where i runs from $n + 1$ to $n + m$ and k runs from 1 to s, be the

[1] Two recent examples are L. R. Klein: "Theories of Effective Demand and Employment," *Journal of Political Economy*, Vol. LV, No. 2, April, 1947; and E. D. Domar: "Capital Accumulation and the End of Prosperity" (Abstract), *Econometrica*, Vol. 16, No. 1, January, 1948.

[2] No attempt will be made to reproduce the process by which the formal definitions were distilled from the sources. The doubting reader will have to resort to a comparison with original works, not limiting himself to the few references given in connection with specific points. Acknowledgment is made to the late Leo Rogin, to S. S. Pu, to D. Hawkins, to P. Sweezy, and to members of the Econometric Society, who made revealing comments on a preliminary form of this paper.

[3] See Eric Roll: *A History of Economic Thought*, pp. 161–75.

amount of the ith product which is consumed in the aggregate by employed labour households of the kth type.[1]

We now define $n + m + s$ quantities t_i by the following equations :

$$t_i q_i = \sum_{j=1}^{n} t_j q^i_j + \sum_{k=1}^{s} t_{n+m+k} h^i_k \qquad i = 1, \ldots n + m \quad \ldots\ldots\ldots\ldots (1)$$

$$t_{n+m+k} h_k = \sum_{j=1}^{m} t_{n+j} q^k_{n+j} \qquad k = 1, \ldots s \quad \ldots\ldots\ldots\ldots (2)$$

$$t_{n+m+1} = 1 \quad \ldots\ldots\ldots\ldots (3)$$

We consider t_{n+m+1} to be without units, so that each term of the equations (1) and (2) has units of labour of type 1. We call $t_i q_i$ " the value of q_i " and $t_{n+m+k} h_k$ " the value of h_k." The quantities t_i have units of labour of type 1 divided by quantity of product or of labour, the latter if i is greater than $n + m$. They might be called " unit values " and are evidently prices of some sort.

In fact, the t_i are those prices which assure zero profits in the accounting balance of each product and the equality of income and consumption expenditure for labour. The third equation merely establishes a numéraire. " Value," so defined, does correspond to the intuitive idea of the amount of labour time chargeable to a product, and it is also evident that no infinite regression is involved, but merely the perfectly familiar and legitimate circularity of simultaneous equations. In an economy of small producers, a plausibility argument may be given to show that goods will actually exchange according to the t's. In fact, if producers can easily shift to different types of production, a price different from the value given by equations (1)–(3) will simply result in one party to the exchange making himself what he might otherwise obtain by exchange.

Thus the equations defining values may be considered as equilibrium conditions of a special type. However, the equations do not form a general equilibrium model since quantities are not determined. They appear, of course, as part of familiar equilibrium models, the bias toward labour being eliminated by the deletion of (3) and a more symmetrical formulation of (1) and (2).[2] If the coefficients of production (q^i_j/q_i and h^i_k/q_i) and of consumption (q^k_{n+j}/h_k) are fixed, and it is assumed that all outputs are used up, we get a zero profit stationary model which determines ratios of prices and quantities.[3]

It appears that in its formal aspects the classical value concept is linked to contemporary ideas in a manner which has been hidden by the smoke of controversy over its normative and political implications. The weakness in the labour theory of value as first formulated by Smith lay, not in any alleged logical inadequacy, but simply in its inapplicability to an economy based on positive profits. The difficulty may be seen by adding together the equations in (1) and in (2). We get :

$$\sum_{i=1}^{n+m} t_i q_i - \sum_{j=1}^{n} t_j \left(\sum_{i=1}^{n+m} q^i_j\right) = \sum_{k=1}^{s} t_{n+m+k} h_k \quad \ldots\ldots\ldots\ldots (1')$$

and

$$\sum_{k=1}^{s} t_{n+m+k} h_k = \sum_{j=1}^{m} t_{n+j} \left(\sum_{k=1}^{s} q^k_{n+j}\right) \quad \ldots\ldots\ldots\ldots (2')$$

[1] Because of the ranges of the sub- and super-scripts, there is no danger of confusing q^k_i and q^i_j.

[2] For example, they appear in the equilibrium model of Walras in his *Elements d'économie politique pure.*

[3] Such a model was constructed by W. W. Leontief as a preliminary to more realistic input-output models. See his " Interrelation of Prices, Output, Savings, and Investment," *Review of Economic Statistics*, Vol. XIX, No. 3, August, 1937, Sections I–IV.

The first of these equations says that the value of labour equals the value of the net product. Thus aggregate non-labour income is zero.[1] Since Smith categorically rejected the sophistry that profits are merely another type of wage,[2] he was obliged to modify the scheme, but neither he nor Ricardo was able to develop a consistent labour theory of value compatible with the existence of profit, although Ricardo was aware of the distinction between the amount of labour which a good can command in exchange and the amount of labour required in its production, a distinction which was used by Marx in his reformulation.[3]

Before passing on to this development, it is worth noting that equations (1′) and (2′) do not make much economic sense except under stationary assumptions. Re-writing the left side of (1′) and noting that it must be equal to the right side of (2′), we have:

$$\sum_{j=1}^{m} t_n + jq_n + j + \sum_{j=1}^{n} t_j \left(q_j - \sum_{j=1}^{n+m} q^i_j\right) = \sum_{j=1}^{m} t_n + j \left(\sum_{k=1}^{s} q^k_n + j\right) \qquad (4)$$

The first term is the value of all consumption goods produced, the second the value of net production of capital goods, the third the value of goods consumed by labour. Under static assumptions, i.e. $\sum_{j=1}^{n+m} q^j_i = q_i$ and $\sum_{k=1}^{s} q^k_i = q_i$, this reduces to an identity. That is satisfactory since the equations (1)–(2) will have a non-trivial solution for the t's only if they are dependent, and in this case the equation (4) would establish this dependence. If, however, there is net accumulation, equation (4) is the paradoxical statement that the value of labour's consumption is greater than the total value of consumption goods produced. In order to make the schema satisfactory under such assumptions we would have to include on the right side of (2) terms involving capital goods. Evidently in a zero-profit expanding economy labour must buy capital goods. However, we are interested in stating Adam Smith's schema rather than in improving it, so we pass on with the remark that (1)–(3) is applicable only to a stationary economy.

III. THE VALUE CONCEPT OF *CAPITAL* VOLUME ONE

The value concept of this section is defined in terms of homogeneous labour time. The hours of homogeneous labour directly involved in producing q_i will be defined by $h^i = \sum_{k=1}^{s} b_k h^i_k$, where the b_k are somehow specified outside the formal structure.[4] Total labour time will be given by $h = \sum_{1}^{n+m} h^i$ and is by definition equal to total value.

[1] " In that original state of things, which precedes both the appropriation of land and the accumulation of stock, the whole produce of labour belongs to the labourer." (*Wealth of Nations*, second sentence of Chapter VIII).

[2] " The profits of stock, it may perhaps be thought, are only a different name for the wages of a particular sort of labour, the labour of inspection and direction. They are, however, altogether different, are regulated by quite sufficient principles, and bear no proportion to the quantity, the hardship, or the ingenuity of this supposed labour of inspection and direction." (*Ibid.*, Ch. VI).

[3] See David Ricardo : *The Principles of Political Economy and Taxation*, Ch. I, Sec. 1.

[4] " Skilled labour counts only as simple labour intensified, or rather multiplied simple labour. . . . The different proportions in which different sorts of labour are reduced to unskilled labour as their standard, are established by a social process that goes on behind the backs of the producers, and consequently, appear fixed by custom. For simplicity's sake we shall henceforth account every kind of labour to be unskilled, simple labour ; by this we do no more than save ourselves the trouble of making the reduction." *Capital*, Vol. I, pp. 51–2. The determination of the b's would be an interesting matter for discussion, but Marx gives almost no indications, and for the present we are limiting ourselves to exposition.

We introduce a new set of quantities associated with labour, N_k where k runs from 1 to s. These represent the number of workers ("labour power") of type k. Then the relation $w_k N_k = h_k$ defines w_k as the average number of hours per man per time interval. We take $N = \sum_{1}^{s} N_k$, $wN = h$, and $q^N{}_j = \sum_{k=1}^{s} q^k{}_j$ $(j = n + 1, \ldots n + m)$. The total labour force is given by N, the number of man-hours per worker by w, and the total amount of the jth product in labour's aggregate budget by $q^N{}_j$.

We now define a set of unit values u_i $(i = 1, \ldots n + m)$ corresponding to the products and a u corresponding to labour power by means of the following equations:

$$u_i q_i = \sum_{j=1}^{n} u_j q^i{}_j + h^i \qquad i = 1, \ldots n + m \quad \ldots \ldots (5)$$

$$uN = \sum_{j=n+1}^{n+m} u_j q^N{}_j \quad \ldots \ldots (6)$$

We will call uN the "value of labour power" and $u_i q_i$ the "value" of q_i. The units of these quantities are those of homogeneous labour hours.[1] That these definitions agree with the intuitive idea of labour time required for (or chargeable to) the production of unit output may be more easily seen if we re-write (5) as:

$$u_i = \sum_{j=1}^{n} u_j \ (q^i{}_j / q_i) + h^i / q_i \qquad i = 1, \ldots n + m \quad \ldots \ldots (5')$$

Here we have unit value as the direct labour time per unit product plus the value of other factors per unit product. With fixed technical coefficients, the equations (5) or (5′) determine the u_i. Then (6) gives u on the assumption that the $q^N{}_j$ in labour's aggregate budget are somehow determined outside the theory.[2]

If we add together equations (5) over all i, we get after collecting terms around the u_i:

$$\sum_{i=1}^{n} u_i \ (q_i - \sum_{j=1}^{n+m} q^j{}_i) + \sum_{i=m+1}^{n+m} u_i q_i = \sum_{i=1}^{n+m} h^i \quad \ldots \ldots (7)$$

The first term of the left member is the value of net output of capital goods, the second is value of consumption goods. The right member is the total labour hours. The

[1] "The labour . . . that forms the substance of value, is homogeneous human labour, expenditure of one uniform labour-power. The total labour-power of society, which is embodied in the sum total of the values of all commodities produced by that society, counts here as one homogeneous mass. . . . The labour-time socially necessary is that required to produce an article under the normal conditions of production, and with the average degree of skill and intensity prevalent at the time. . . . We see then that that which determines the magnitude of the value of an article is the amount of labour socially necessary, or the labour-time socially necessary for its production. Each individual commodity, in this connection, is to be considered as an average sample of its class." *Ibid.*, pp. 45–6.

[2] This does not, of course, require a "subsistence" wage theory unless one sets the $q^N{}_j$ at such a level. The following quotation indicates how wide of the mark have been some of the criticisms based on the illusion that the theory implies a minimum level of subsistence: "The value of labour-power is determined, as in the case of every other commodity, by the labour-time necessary for the production, and consequently also the reproduction, of this special article. . . . Given the individual, the production of labour-power consists in his reproduction of himself or his maintenance. For his maintenance he requires a given quantity of the means of subsistence. Therefore the labour-time requisite for the production of labour-power reduces itself to that necessary for the production of those means of subsistence. . . . His means of subsistence must therefore be sufficient to maintain him in his normal state as a labouring individual. . . . On the other hand, the number and extent of his so-called necessary wants, as also, the modes of satisfying them, are themselves the product of historical development, and depend therefore to a great extent on the degree of civilization of a country, more particularly on the conditions under which, and consequently on the habits and degree of comfort in which, the class of free labourers has been formed. In contradistinction therefore to the case of other commodities, there enters into the determination of the value of labour-power a historical and moral element. Nevertheless, in a given country, at a given period, the average quantity of the means of subsistence necessary for the labourer is practically known." *Ibid.*, pp. 189–90.

equation says that value of net output equals total labour time. It may be compared with (1′) and (4), whose left members are the same if the t's be replaced by u's, but whose right members are the " value of labour," i.e. labour income, whereas the right member of (7) is merely the total labour time. Since labour's income uN is not necessarily equal to the right member of (7), which we call h, there is the possibility of non-labour income. If we write $S = h - uN = (w - u)\ N$, S is " surplus value," h is " value," uN is " variable capital," and S/uN is the " rate of surplus value." " Surplus value " is thus the value of the products which are neither used up in production nor consumed by labour.

The careful reader will have noted that we have so far made no statements about prices. It remains to be seen whether value so defined is related to price in any significant way. To begin with, if we consider a " simple commodity production " economy, the surplus will be zero and the same argument would apply as we mentioned in connection with Adam Smith's schema. In fact, in this case, the value schema reduces to that of Smith. That Smith's definition is a special case of Marx's is interesting, but the latter was specifically developed to take care of a profit situation so that it is not too helpful to know that it applies to a zero profit economy. Marx maintained that value as here defined was relevant to studying aggregate or typical exchange relations, and in Vols. I and II of *Capital*, where he assumes that goods do exchange according to their values, the assumption appears to be equivalent to an aggregation or averaging.[1] It has also been held, although not explicitly by Marx, that value is a normative concept, representing the way in which goods should exchange. The dominant point of view among Soviet economists is that price should deviate from value only in a planned purposeful fashion, but the precise relationship between value and price in a socialist economy is still a matter of controversy.[2] In any case it is clear that the u's give some information as to the labour time chargeable to each product, taking into account indirect as well as direct labour. Equations (5) and (6) could be used for *ex-post* accounting or as part of planning computations with virtual quantities. Whether or not it would be desirable to set prices proportional to the u's, they are a relevant datum in an economy.

Whatever the relation between value and price in a planned economy, it is easy to see that under conditions of capital mobility in a competitive economy goods will not exchange according to their values. In order to show this we assume that goods do exchange according to their values and consider the balance sheet of an industry. Since we are dealing with homogeneous labour, we assume that the industry pays for this at the average hourly wage rate, i.e. $uN/h = u/w$. Then the profits of the ith industry will be given by:

$$\pi_i = u_i q_i - \sum_{j=1}^{n} u_j q^i_j - \frac{u}{w} h^i \qquad (8)$$

Replacing $u_i q_i$ by its equivalent from (5), we have:

$$\pi_i = h^i - \frac{u}{w} h^i$$

[1] In order to justify this interpretation it would be necessary to work out fully the implications of the above definitions in terms of aggregates and show that the resulting structure was equivalent to that of Vols. I and II and also independent of deviations of individual prices and values. The task is too big for this article, but it is easy to find quotations to show that Marx explicitly limited himself to aggregate or typical relations. For example, see *Capital*, Vol. I, pp. 46, 221, 332, 671–672.

[2] The Russian literature is large. Two illuminating translations are " Political Economy in the Soviet Union," by L. A. Leontiev and others, *Science and Society*, Vol. VIII, No. 2, Spring, 1944; and " Basic Laws of Development of Socialist Economy," by K. Ostrovitianov, *Science and Society*, Vol. IX, No. 3, Summer, 1945.

$$= \left(\frac{w}{u} - 1\right) \frac{u}{w} h^i$$

$$= \left(\frac{w - u}{u}\right) \frac{u}{w} h^i$$

$$= \sigma \frac{u}{w} h^i \quad \ldots\ldots (9)$$

where σ is the "rate of surplus value" defined as S/uN, since $(w - u)/u = (wN - uN)/uN = (h - uN)/uN$. Now σ is independent of i, and uh^i/w is the wage bill ("variable capital"). It follows that profits are proportional to the wage bill, a conclusion which is most inconsistent with experience. Equation (9) would require different profit rates in different industries, a condition evidently incompatible with equilibrium. Thus prices must actually differ in a systematic way from values in a competitive economy.[1] Such considerations led Marx to the "price of production" concept, which we will consider in the next section.

III. PRICE OF PRODUCTION

The notion of price of production is as old as economics itself. It is based on the observation that the rate of profit tends to be uniform over all industries.[2] We define a rate of profit r and unit prices of production x_i and x by means of the following equations:

$$x_i q_i = (1 + r) \left(\sum_{j=1}^{n} x_j q^i_j + \frac{x h^i}{w} \right) \qquad i = 1, \ldots n + m \quad \ldots\ldots (10)$$

$$xN = \sum_{j=n+1}^{n+m} x_j q^N_j \quad \ldots\ldots (11)$$

$$\sum_{i=1}^{n} x_i \left(q_i - \sum_{j=1}^{n+m} q^j_i\right) + \sum_{i=n+1}^{n+m} x_i q_i = h \quad \ldots\ldots (12)$$

The first of these equations states that the price of production of the ith product is equal to one plus the profit rate times total outlay.[3] The second is the familiar equality of labour income and expenditure. The third is characteristic of Marx's treatment. It assures that the sum of values equals the sum of the prices of production, so that this price of production schema amounts to a different imputation of the total labour time to the different commodities. The units of prices of production are the same as of values, labour hours. Without (12), equations (10) and (11) determine merely the ratios of the prices of production.

Equations (10)–(12) do not constitute a complete equilibrium system, but (10) gives necessary conditions for equilibrium under capital mobility, and (11) is necessary for a balance of income and expenditure by labour. Hence, it seems reasonable to conclude that there is an actual tendency for exchange to take place at prices satis-

[1] The argument is a mathematical formulation of that given in Vol. I of *Capital*, chapter XI.

[2] "We call it price of production. It is, as a matter of fact, the same thing which Adam Smith called *natural price*, Ricardo *price of production*, or *cost of production*, and the physiocrats *prix necessaire*, because it is in the long run a prerequisite of supply. . . ." *Capital*, Vol. III, p. 233. (Italics in original.)

[3] Strictly speaking the profit should be calculated on total invested capital instead of on current outlay plus depreciation as it is here, but this is the procedure which Marx follows. It is equivalent to taking the time interval to coincide with a period of turnover of fixed capital.

fying (10) and (11), the quantities being determined in some manner not accounted for in the value schema. If, however, we assume given technical coefficients (q^i_j/q_i, h^i/q_i, q^N_j/N, and w), the system is complete. Because of its linear character it is well adapted to the investigation of " technological change " as reflected in changes in these technical coefficients.[1]

It has not been maintained, of course, that prices are always equal to prices of production, but Marx did maintain that the price of production of an industry was an average about which actual prices were distributed both in time and from one firm to another.[2] More recently, it has been claimed that price of production should play a normative role as the correct price in a socialist economy.[3] Whatever the merits of this position, it is clear that setting prices according to prices of production is a possible procedure and one which allows for a surplus. If it is desired to set prices so that the value of " surplus " production (i.e. production not consumed by labour) is charged against each product in proportion to cost, then price of production is appropriate. If one wishes to charge the surplus to products in proportion to direct labour cost, then value is the proper pricing norm.

It appears from our discussion that the classical value schemata are neither meaningless nor very startling. Perhaps the formal definitions which we have given might be used to clarify more than one of the controversies that have raged around the labour theory stated in vaguer terms, but we will apply them here only to the so-called " transformation problem." Stated briefly, the problem is this: In Vol. I, values and prices are assumed proportional. In Vol. III it is shown that prices tend to proportionality with prices of production. Yet Marx asserts that value determines price. In order to justify this it is necessary to show that the values of a given constellation of products and factors determine the corresponding prices of production. Hence the problem of " transforming " values into prices of production.[4] In terms of the definitions of this paper, the problem might be posed as one of deriving the x's from the u's. But actually the question has not been formulated in this way, and understandably so, since a glance at equations (5)–(6) and (10)–(12) indicates that the derivation is not generally possible. Instead it has been asked whether it can be shown that the $u_i q_i$ and uN determine the $x_i q_i$ and xN. In order to state the matter briefly, we define $c^i_j = u_j q^i_j$, $C^i_j = x_j q^i_j$, $v_i = uh^i/w$, $V_i = xh^i/w$, $a_i = u_i q_i$, $A_i = x_i q_i$,

[1] Somewhat similar equations were utilised by Leontief for this purpose in the article cited in Section 2 above, in other articles, and in his *The Structure of the American Economy*, Cambridge, Mass., 1941.

[2] See Chapter X of Vol. III of *Capital*, where the relation between market price and value is discussed with the remark that the same considerations apply " with the necessary modifications to the price of production."

[3] Ch. Bettelheim : *Les Problemes Theoriques et Pratique de la Planification*, Paris, 1946, esp. pp. 1–12, 183–215. This is, as far as the writer is aware, the only non-Soviet effort to relate the labour theory of value to planning practice.

[4] The question was posed in this manner by Bortkiewicz in a paper in the *Jahrbücher für Nationalökonomie und Statistik*, July, 1907. Paul M. Sweezy reproduced the argument in his *The Theory of Capitalist Development*, New York, 1942, and J. Winternitz clarified the procedure in " Values and Prices," *Economic Journal*, 1948, pp. 276 *et seq*. The transformation problem is an off-shoot of the famous controversy over the alleged " contradiction " between Vol. I and III. The classic on that subject is Bohm-Bawerk's *Karl Marx and the Close of His System*. An interesting summary is to be found in William J. Blake, *An American Looks at Karl Marx*, New York, 1939. In passing, we mention the error (current in vulgarizations of the controversy) that the price of production theory was invented to reconcile an unforeseen contradiction between the supposed belief that goods do exchange at their values and the ubiquitous evidence that they do not. Actually, the theory was outlined by Marx in a letter to Engels, of August 2nd, 1862, and Engels states in the preface to Vol. II of *Capital* that Marx had worked it out in his manuscript for the *Critique of Political Economy*, which was published in 1859. Vol. I of *Capital* was not published until 1867. There Marx distinguished from the beginning between value and price, pointing out that they would coincide only under special circumstances and that a good might have a price without having a value at all (p. 115). He mentions the " apparent contradiction " between the assumption of exchange at values and actual exchanges and indicates that a solution will be given after the necessary " intermediate terms " have been worked out (p. 335).

$s_i = a_i - \sum_{j=1}^{n} c^i_j - v_i$, $S_i = A_i - \sum_{j=1}^{n} C^i_j - V_i$. Then for the value schema we have:

$$\sum_{j=1}^{n} c^i_j + v_i + s_i = a_i \qquad i = 1, \ldots n + m^1 \quad (13)$$

For prices of production we have similarly:

$$\sum_{j=1}^{n} C^i_j + V_i + S_i = A_i \qquad i = 1, \ldots n + m \quad (14)$$

Now it follows from the definitions above that $C^i_j/c^i_j = A_j/a_j = x_j/u_j = \lambda_j$ and $V_i/v_i = x/u = \lambda$. If we can determine the λ's as functions of the c's, v's, s's and a's, we will have solved the " transformation problem." From the definitions for the λ's and from (10) it follows that (14) may be written:

$$\left(\sum_{j=1}^{n} \lambda_j c^i_j + \lambda v_i\right) (1 + r) = \lambda_i a_i \qquad i = 1, \ldots n + m \quad (15)$$

From (6) and (11) it follows that:

$$\lambda = \left(\sum_{j=n+1}^{n+m} \lambda_j c^N_j\right) / \left(\sum_{j=n+1}^{n+m} c^N_j\right) \quad (16)$$

where by definition c^N_j is $u_j q^N_j$. Finally, from (12) we have:

$$\sum_{i=1}^{n+m} \lambda_i a_i = \sum_{i=1}^{n+m} a_i \quad (17)$$

These $n + m + 2$ equations involve $n + m + 1$ λ's and the profit rate r. Hence they can be solved for these quantities in terms of the values.

The " transformation problem " is, at least in this form, seen to be merely a formal matter. Most of the complications in the controversy seem to be due to differences over interpretation and a tendency to confuse this formal problem of relating values and prices of production of given quantities with the much more complicated question of how a situation of exchange at values evolves into one of exchange at prices of production.[2] The above general solution enables us to clarify a difficulty which arises from placing the problem in the context of three industries. If in the above schema we take $n = 1$, $m = 2$, $q^N_2 = q_2$, $q^N_3 = 0$, we get a three industry model in which industry 1 produces capital goods, industry 2 produces wage goods, and industry 3 produces " luxury " goods. Our equations reduce to the following:

$$c^1_1 + v_1 + s_1 = a_1 \quad (13')$$
$$c^2_1 + v_2 + s_2 = a_2$$
$$c^3_1 + v_3 + s_3 = a_3$$

$$(\lambda_1 c^1_1 + \lambda_2 v_1)\ (1 + r) = \lambda_1 a_1 \quad (15')$$
$$(\lambda_1 c^2_1 + \lambda_2 v_2)\ (1 + r) = \lambda_2 a_2$$
$$(\lambda_1 c^3_1 + \lambda_2 v_3)\ (1 + r) = \lambda_3 a_3$$

[1] From (5) and (9) it is clear that $s_i = \sigma v_i$, and there may, of course, be other relationships among the quantities in (13), but these do not concern us here. However, the gratuitous assumption that the quantities in any one of the $n + m$ equations are independent of those in the others has been the basis of dubious reasoning by Bortkiewicz and others. In this connection, see Kenneth May, " Value and Price of Production: A Note on Winternitz' Solution," *Economic Journal*, Vol. LVIII, No. 232, December, 1948, pp. 596–99.

[2] See the articles previously cited and the writer's comments in the note mentioned above.

$$\lambda = \lambda_2 \ldots\ldots (16')$$

$$a_1 + a_2 + a_3 = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3. \ldots\ldots (17')$$

This is the solution given by Winternitz. It serves as an example, or special case, of the general solution. If, however, it is interpreted as a solution in terms of three industries aggregated from the complete economy, then it can hardly be considered satisfactory.

Let us try to derive (13')–(17') by aggregating (13)–(17). If we add the equations (14) over the n capital goods industries, over, say, p wage goods industries, and over the remaining m–p luxury goods industries we get:

$$\bar{c}_i + v_i + \bar{s}_i = a_i \qquad i = 1, 2, 3 \ldots\ldots (13'')$$

where $\bar{c}_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} c^i_j$, $v_1 = \sum_{i=1}^{n} v_i$, $\bar{s}_1 = \sum_{i=1}^{n} s_i$, $\bar{a}_1 = \sum_{i=1}^{n} a_i$, and similarly for other values of i. In the same way we can write:

$$\bar{C}_i + \bar{V}_i + \bar{S}_i = \bar{A}_i \qquad i = 1, 2, 3 \ldots\ldots (14'')$$

where the aggregates are defined as above, e.g. $\bar{C}_1 = \sum_{i=1}^{n} \sum_{i=1}^{n} C^i_j$. We can also write an equation corresponding to (15), since the $(1 + r)$ runs through all equations. Adding the equations (15) over appropriate ranges gives:

$$(\bar{C}_i + \bar{V}_i)\ (1 + r) = \bar{A}_i \qquad i = 1, 2, 3 \ldots\ldots (15'')$$

But now we run into difficulties. We are going to come out, or we wish to come out, with five equations involving r and the four λ's. But it is not true that the C's, V's and A's can be expressed in terms of the c's, v's and a's, and only four λ's. We do not know that for example that $\bar{C}_i/\bar{c}_i$ is the same for all i. In fact:

$$\bar{C}_1/\bar{c}_1 = \frac{\sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} c^i_j}{\sum_{j=1}^{n} \sum_{i=1}^{n} c^i_j}$$

and

$$\bar{C}_2/\bar{c}_2 = \frac{\sum_{j=1}^{n} \lambda_j \sum_{i=n+1}^{n+p} c^i_j}{\sum_{j=1}^{n} \sum_{i=n+1}^{n+p} c^i_j}$$

There is no reason to think that these should be equal, although they are averages of the λ's and therefore may not differ very much. Still we are not justified in writing (15'') in the form (15'), and the model (13')–(17') is only an approximation when the variables are interpreted as aggregates.

It appears that the transformation problem is not solvable in terms of aggregates alone. Individual commodities must be considered because they enter into the production of different industries in different proportions and hence the constant capitals (the c's and C's) involved in aggregate industries have prices of production and values in different ratios in different industries. The result provides another example of the importance of distinguishing between a simplified model obtained as a special case and those obtained by deriving relations between aggregates. Discussion of the aggregation problem has shown that one cannot count on the two being of the same form.[1]

Northfield, Minnesota. KENNETH MAY.

[1] See André Nataf, "Sur la Possibilité de Construction de Certain Macromodéles," *Econometrica*, Vol. 16, No. 3, July, 1948, pp. 232–44, and the references there cited.